Existence and Blow up of Solutions of the Cauchy Problem of the Generalized Damped Multidimensional Improved Modified Boussinesq Equation

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We consider the existence, both locally and globally in time, and the blow up of solutions of the Cauchy problem of the generalized damped multidimensional improved modified Boussinesq equation in $W^{s,p}(\mathrm{R}^n)$.

Key words: Damped Improved Modified Boussinesq Equation; Cauchy Problem; Local Solution; Global Solution; Blow up of Solution.

1. Introduction

We study the Cauchy problem of the generalized damped multidimensional improved modified Boussinesq (IMBq) equation

$$u_{tt} - \Delta u - \Delta u_{tt} - \Delta u_t = \Delta f(u),\quad (x,t) \in \mathrm{R}^n \times (0,+\infty), \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_t(x), \quad x \in \mathrm{R}^n, \tag{2}$$

where $u(x,t)$ denotes the unknown function, $f(s)$ is the given nonlinear function, $u_0(x)$ and $u_1(x)$ are the given initial value functions, the subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x, and Δ denotes the Laplace operator in R^n.

Scott Russell's study [1] of solitary water waves motivated the development of nonlinear partial differential equations for the modeling wave phenomena in fluids, plasmas, elastic bodies, etc. It is well known that the Boussinesq equation can be written in two basic forms:

$$u_{tt} - u_{xx} + \delta u_{xxxx} = (u^2)_{xx}, \tag{3}$$

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}. \tag{4}$$

Equation (4) is an important model that approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxxx} instead of u_{xxtt}). In the case of $\delta > 0$ (3) is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (see [2] and references therein). It is called the "good" Boussinesq equation, while the equation with $\delta < 0$ received the name "bad" Boussinesq equation since it possesses linear instability. Equation (3) was first deduced by Boussinesq [3]. Equation (4) is called improved Boussinesq (IBq) equation.

There is a considerable mathematical interest in the Boussinesq equations which have been studied from various aspects (see [4 – 7] and references therein). A great deal of efforts has been made to establish sufficient conditions for the nonexistence of global solutions to various associated boundary value problems [6, 8]. Levine and Sleeman [8] studied the global nonexistence of solutions for the equation

$$u_{tt} - u_{xx} - 3u_{xxxx} + 12(u^2)_{xx} = 0$$

with periodic boundary conditions. Turitsyn [6] proved the blow up in the Boussinesq equations

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0$$

and

$$u_{tt} - u_{xx} - u_{xxtt} + (u^2)_{xx} = 0$$

for the case of periodic boundary conditions and obtained exact sufficient criteria of the collapse dynamics.

The generalization of the Boussinesq equation was done in numerous studies [9 – 18]. Liu [13, 14] studied the instability of solitary waves and the existence, both locally and globally in time, of the generalized Boussinesq-type equation

$$u_{tt} - u_{xx} + (f(u) + u_{xx})_{xx} = 0,$$

and established some blow up results of the nonlinear Pochhammer-Chree equation

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0. \quad (5)$$

Godefroy [9] showed the blow up of the solutions of the Cauchy problem for (5) and he focused on various perturbations of the equation. Guowang and Shubin [10] proved the existence and nonexistence of a global solution of the generalized IMBq equation

$$u_{tt} - u_{xx} - u_{xxtt} = f(u)_{xx}.$$

Zhijian [17] and Yang and Wang [18] studied, respectively, the existence and blow up of solutions to the initial boundary value problems of the generalized Boussinesq equations

$$u_{tt} - u_{xx} - bu_{xxxx} = \sigma(u)_{xx}$$

and

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx}.$$

Makhankov [19] pointed out that the IBq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^2)$$

can be obtained by starting with the exact hydrodynamical set of equations in plasma, and a modification of the IBq equation, analogous to the modified Korteweg-de Vries equation, yields

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^3), \quad (6)$$

which is the so-called IMBq (modified IBq) equation.

Wang and Chen [20, 21] studied the existence, both locally and globally in time, and nonexistence of solutions, and the global existence of small amplitude solutions of the Cauchy problem of the multidimensional generalized IMBq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta f(u). \quad (7)$$

In the Boussinesq equations, the effects of small nonlinearity and dispersion are taken into consideration, but in many real situations, damping effects are compared in strength to nonlinear and dispersive ones. Therefore the damped Boussinesq equations is considered as

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad (8)$$

where u_{txx} is the damping term and $\alpha, b = \text{const} > 0$, $\beta = \text{const} \in R$ (see [2, 4, 5, 7, 22] and references therein).

Varlamov [2, 7] investigated the long-time behaviour of solutions to an initial value, spatially periodic, and initial-boundary value problems of (8) in two space dimensions. Polat et al. [15] established the blow up of solutions for the initial boundary value problem of the damped Boussinesq equation

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = f(u)_{xx}.$$

Lai and Wu [4] and Lai et al. [5] investigated, respectively, the global solution of the following generalized damped Boussinesq equations:

$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} - p^2u + \beta(u^2)_{xx},$$
$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} + \beta(u^2)_{xx}. \quad (9)$$

Polat and Kaya [16] established the blow up of the solutions for the initial boundary value problem of (9).

Our goal in this paper is to extend the result of [20] to the damped version of the problem (1) and (2). First, by using the contraction mapping principle, we establish the locally well posedness of the Cauchy problem. Then we derive the necessary a priori bounds that guarantee that every local solution is indeed global in time. Finally, we discuss the local solution of the Cauchy problem with negative and nonnegative initial energy blow up in finite time by using the concavity method.

Throughout this paper, we use the following notations and lemmas: L^p denotes the usual space of all L^p functions on R^n with norm $\|f\|_{L^p} = \|f\|_p$; $W^{s,p}$ denotes the usual Sobolev space on R^n with norm $\|f\|_{s,p} = \sum_{k=0}^{s} \|D^k f\|_p$, where $\|D^k f\|_p = \sum_{|\alpha|=k} \|D^k f\|_p$, s is a positive integer, $1 \le p \le \infty$, and

$$D^k u = \left\{ \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} : |\alpha| = \sum_{i=1}^{n} \alpha_i = k, \quad \alpha_i \ge 0 \quad (i = 1, 2, \cdots, n) \right\}.$$

Lemma 1 (Nirenberg's inequality) [23]. Suppose that $u \in L^p$, $D^m u \in L^q$, $1 \le p, q \le \infty$. Then for any i $(0 \le i \le m)$, we have

$$\|D^i u\|_r \le C \|u\|_p^{1-i/m} \|D^m u\|_q^{i/m},$$

where

$$\frac{1}{r}=\left(1-\frac{i}{m}\right)\frac{1}{p}+\frac{i}{m}\frac{1}{q},$$

and C is a constant independent of u.

Using the chain rule of the composite function, we can prove the following result from Lemma 1.

Lemma 2 [24]. Suppose that $u\in W^{s,p}\cap L^{\infty}$, and $f(u)$ possesses continuous derivatives up to order $s\geq 1$. Then $f(u)-f(0)\in W^{s,p}$ and

$$\|f(u)-f(0)\|_p\leq \|f'(u)\|_{\infty}\|u\|_p,$$

$$\|D^k f(u)\|_p\leq C_0\sum_{p=1}^{k}\left(\|f^p(u)\|_{\infty}\|u\|_{\infty}^{p-1}\right)\|D^k u\|_p \quad (1\leq k\leq s),$$

where $C_0\geq 1$ is a constant.

Lemma 3 (Minkowski's inequality for integrals) [25]. If $1\leq p\leq\infty$, $u(x,t)\in L^p(\mathrm{R}^n)$ for a.e. t, and function $t\rightarrow\|u(\cdot,t)\|_p$ is in $L^1(I)$, where $I\subset[0,\infty)$ is an interval, then

$$\left\|\int_I u(\cdot,t)\mathrm{d}t\right\|_p\leq\int_I\|u(\cdot,t)\|_p\mathrm{d}t.$$

The plan of this paper is as follows. In Section 2, we study the existence and uniqueness of the local solutions for problem (1) and (2). The global well posedness of the problem is given in Section 3. In Section 4, we discuss the blow up of solution of the problem.

2. Existence and Uniqueness of the Local Solution

In this section, we prove the existence and the uniqueness of the local solution for problem (1) and (2) by the contraction mapping principle.

For this purpose, let $G(x)$ (see [26, 27]) be the fundamental solution of the partial differential equation

$$w(x)-\Delta w(x)=0. \tag{10}$$

By the use of Fourier transform, we obtain

$$G(x)=\frac{1}{(4\pi)^{n/2}}\int_0^{\infty}\mathrm{e}^{-|x|^2/4\delta}\mathrm{e}^{-\delta}\delta^{-n/2}\mathrm{d}\delta,\quad x\in\mathrm{R}^n. \tag{11}$$

The fundamental solution $G(x)$ satisfies the following properties in Lemma 4.

Lemma 4. (i) $G(x)$ is defined and continuous on R^n, and $G(x)>0$.

(ii) $G(x)$ satisfies the equation

$$G(x)-\Delta G(x)=\delta(x),$$

where $\delta(x)$ is the Dirac delta function.

(iii) $G(x)\in L^q(\mathrm{R}^n)$ and $\|G(x)\|_1=1$, where

$$\begin{aligned}&1\leq q\leq\infty,\ \text{if } n=1,\\ &1\leq q<\infty,\ \text{if } n=2,\\ &1\leq q<\frac{n}{n-2},\ \text{if } n\geq 3.\end{aligned} \tag{12}$$

Proof. For the proof of (i) and (ii), see [26], and we give reference [20] for the proof of (iii).

Suppose that $u(x,t)\in\mathrm{C}^2([0,T];W^{2,p}\cap L^{\infty})$ is a solution of problem (1) and (2). We can rewrite (1) as follows:

$$\begin{aligned}&[u_{tt}+u+u_t+f(u)]-\Delta[u_{tt}+u+u_t+f(u)]\\ &\quad=u+u_t+f(u).\end{aligned} \tag{13}$$

For the sake of convenience, we assume that $f(0)=0$. Otherwise we can replace $f(u)$ with $f(u)-f(0)$. From Lemma 2 we have $f(u)\in W^{2,p}$, if $f(u)\in\mathrm{C}^2(\mathrm{R})$.

From (10) and (13), we get

$$u_{tt}+u+u_t+f(u)=G*[u+u_t+f(u)], \tag{14}$$

where $u*v$ denotes the convolution of u and v; it is defined by

$$u*v=\int_{\mathrm{R}^n}u(y)v(x-y)\mathrm{d}y.$$

From (2) and (14), we know that the Cauchy problem (1) and (2) is equivalent to the integral equation

$$\begin{aligned}u(x,t)&=u_0(x)\cos t+u_1(x)\sin t\\ &-\int_0^t\sin(t-\tau)\left[u_\tau(x,\tau)+f(u(x,\tau))\right]\mathrm{d}\tau\\ &+\int_0^t\sin(t-\tau)G*\left[u(x,\tau)+u_\tau(x,\tau)+f(u(x,\tau))\right]\mathrm{d}\tau.\end{aligned} \tag{15}$$

Now we are going to prove the existence and the uniqueness of the local solution for the integral equation (15) by the contraction mapping principle.

Let us define the function space

$$X(T)=\mathrm{C}^1([0,T];W^{2,p}\cap L^{\infty}),$$

which endowes with the norm

$$\|u\|_{X(T)} = \max_{0\le t\le T}\|u\|_{2,p} + \max_{0\le t\le T}\|u_t\|_{2,p} + \max_{0\le t\le T}\|u\|_\infty + \max_{0\le t\le T}\|u_t\|_\infty, \quad \forall u \in X(T).$$

It is easy to see that $X(T)$ is a Banach space. For any initial values $u_0, u_1 \in W^{2,p}\cap L^\infty$, let $M = \|u_0\|_{2,p} + \|u_1\|_{2,p} + \|u_0\|_\infty + \|u_1\|_\infty$. Take the set

$$Y(M,T) = \{u \mid u\in X(T), \|u\|_{X(T)} \le 2M+1\}.$$

Obviously, $Y(M,T)$ is a nonempty bounded closed convex subset of $X(T)$ for any fixed $M>0$ and $T>0$. We define the map H as

$$\begin{aligned} Hu(x,t) &= u_0(x)\cos t + u_1(x)\sin t \\ &- \int_0^t \sin(t-\tau)[u_\tau(x,\tau) + f(u(x,\tau))]\mathrm{d}\tau \\ &+ \int_0^t \sin(t-\tau) G*[u(x,\tau)+u_\tau(x,\tau)+f(u(x,\tau))]\mathrm{d}\tau, \end{aligned} \tag{16}$$

where $u\in X(T)$. From Lemma 2, it is easy to see that H is well defined if $f(u)\in \mathrm{C}^2(\mathrm{R})$, and we can easily show that H maps $X(T)$ into $X(T)$. Our goal is to show that H has a unique fixed point in $Y(M,T)$ for suitable T.

Lemma 5. Assume that $u_0, u_1 \in W^{2,p}\cap L^\infty$ and $f(s)\in \mathrm{C}^3(\mathrm{R})$. Then H is contractive mapping from $Y(M,T)$ into itself for T is sufficiently small relative to M.

Proof. We first prove that H maps $Y(M,T)$ into itself for T is small enough. Let $u\in Y(M,T)$ be given. Let us define $\bar f(\eta): [0,\infty)\to[0,\infty)$ by

$$\bar f(\eta) = \left\{|f'(s)|, |f''(s)|, |f'''(s)|\right\}, \quad \forall \eta\ge 0.$$

We observe that $\bar f$ is continuous and nondecreasing on $[0,\infty)$. From Lemma 2 we have

$$\|f(u)\|_{2,p} \le 2\mathrm{C}_0 \bar f(2M+1)(2M+1)\|u\|_{2,p}. \tag{17}$$

Using Young's inequality and Lemma 4, we obtain

$$\begin{aligned} &\|G*(u+u_t+f(u)\|_\infty \le \|u+u_t+f(u)\|_\infty, \\ &\|G*(u+u_t+f(u)\|_{2,p} \le \|u+u_t+f(u)\|_{2,p}. \end{aligned} \tag{18}$$

From (16), Lemma 3 and (18), it follows that

$$\begin{aligned} \|Hu\|_\infty &\le \|u_0\|_\infty + \|u_1\|_\infty + \int_0^t \|u(\tau)\|_\infty \mathrm{d}\tau \\ &+2\int_0^t \|u_\tau(\tau)\|_\infty\mathrm{d}\tau + 2\int_0^t \|f(u(\tau))\|_\infty \mathrm{d}\tau, \end{aligned} \tag{19}$$

$$\begin{aligned} \|Hu_t\|_\infty &\le \|u_0\|_\infty + \|u_1\|_\infty + \int_0^t \|u(\tau)\|_\infty \mathrm{d}\tau \\ &+2\int_0^t \|u_\tau(\tau)\|_\infty\mathrm{d}\tau + 2\int_0^t \|f(u(\tau))\|_\infty \mathrm{d}\tau, \end{aligned} \tag{20}$$

$$\begin{aligned} \|Hu\|_{2,p} &\le \|u_0\|_{2,p} + \|u_1\|_{2,p} + \int_0^t \|u(\tau)\|_{2,p} \mathrm{d}\tau \\ &+2\int_0^t \|u_\tau(\tau)\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|f(u(\tau))\|_{2,p} \mathrm{d}\tau, \end{aligned} \tag{21}$$

$$\begin{aligned} \|Hu_t\|_{2,p} &\le \|u_0\|_{2,p} + \|u_1\|_{2,p} + \int_0^t \|u(\tau)\|_{2,p} \mathrm{d}\tau \\ &+2\int_0^t \|u_\tau(\tau)\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|f(u(\tau))\|_{2,p} \mathrm{d}\tau. \end{aligned} \tag{22}$$

Thus, from (17), (19) – (22), and Lemma 2 we have

$$\|Hu\|_{X(T)} \le 2M + 4(2M+1) \cdot [1+2C_0(2M+1)\bar f(2M+1)]T.$$

If T satisfies

$$T \le \frac{1}{4(2M+1)[1+2C_0(2M+1)\bar f(2M+1)]}, \tag{23}$$

then

$$\|Hu\|_{X(T)} \le 2M+1. \tag{24}$$

Therefore, if condition (24) holds, then H maps $Y(M,T)$ into $Y(M,T)$.

Now we are going to prove that the map H is strictly contractive. Let $T>0$ and $u,v\in Y(M,T)$ be given, then we have

$$\begin{aligned} Hu - Hv = &-\int_0^t \sin(t-\tau)[u_\tau(x,\tau) - v_\tau(x,\tau) + f(u(x,\tau)) - f(v(x,\tau))]\mathrm{d}\tau \\ &+ \int_0^t \sin(t-\tau) G*[u(x,\tau) - v(x,\tau) + u_\tau(x,\tau) - v_\tau(x,\tau) + f(u(x,\tau)) - f(v(x,\tau))]\mathrm{d}\tau. \end{aligned} \tag{25}$$

By means of the mean value theorem, Hölder's inequality and Nirenberg's inequality, we obtain

$$\|f(u)-f(v)\|_\infty \le \bar f(2M+1)\|u-v\|_\infty, \tag{26}$$

$$\|f(u)-f(v)\|_p \le \bar f(2M+1)\|u-v\|_p, \tag{27}$$

$$\|D(f(u)-f(v))\|_p \le \bar f(2M+1)(2M+1)\|u-v\|_\infty + \bar f(2M+1)\|D(u-v)\|_p, \tag{28}$$

$$\begin{aligned}&\|D^2(f(u)-f(v))\|_p\\ &\le 3C^2\bar f(2M+1)(2M+1)^2\|u-v\|_\infty\\ &\quad + 2C^2\bar f(2M+1)\|D^2(u-v)\|_p,\end{aligned} \tag{29}$$

where C is the constant in Lemma 1. From (25) – (29), using Lemma 3, Lemma 4, and Young's inequality, we get

$$\begin{aligned}&\|Hu-Hv\|_\infty+\|(Hu-Hv)_t\|_\infty\\ &\le 2\int_0^t \|u-v\|_\infty \mathrm{d}\tau + 4\int_0^t \|(u-v)_\tau\|_\infty \mathrm{d}\tau\\ &\quad +4\int_0^t \|f(u)-f(v)\|_\infty \mathrm{d}\tau,\end{aligned}$$

$$\begin{aligned}&\|Hu-Hv\|_{2,p}+\|(Hu-Hv)_t\|_{2,p}\\ &\le 2\int_0^t \|u-v\|_{2,p} \mathrm{d}\tau + 4\int_0^t \|(u-v)_\tau\|_{2,p} \mathrm{d}\tau\\ &\quad +4\int_0^t \|f(u)-f(v)\|_{2,p} \mathrm{d}\tau,\end{aligned}$$

$$\|Hu-Hv\|_{X(T)} \le 4[1+4\bar f(2M+1)(1+3C^2(2M+1)^2)]T\|u-v\|_{X(T)}.$$

Take T satisfying (23) and

$$T < \frac{1}{4[1+4\bar f(2M+1)(1+3C^2(2M+1)^2)]}, \tag{30}$$

then

$$\|Hu-Hv\|_{X_k(T)} < \|u-v\|_{X_k(T)}. \tag{31}$$

This shows that $H: Y(M,T) \to Y(M,T)$ is strictly contractive. The lemma is proved.

Theorem 1. Assume that the conditions of Lemma 5 hold, then problem (1), (2) admits a unique local solution $u(x,t) \in C^1([0,T_0); W^{2,p}\cap L^\infty)$, where $[0,T_0)$ is the maximal time interval of existence for $u(x,t)$. Moreover, if

$$\begin{aligned}\sup_{t\in[0,T_0)} \big(&\|u(\cdot,t)\|_{2,p}+\|u_t(\cdot,t)\|_{2,p}\\ &+\|u(\cdot,t)\|_\infty+\|u_t(\cdot,t)\|_\infty\big) < \infty,\end{aligned} \tag{32}$$

then $T_0=\infty$.

Proof. From Lemma 5 and the contraction mapping principle it follows that for appropriately chosen $T>0$, H has a unique fixed point $u(x,t) \in Y(M,T)$, which is a solution of problem (1), (2). It is not difficult to prove the uniqueness of the solution which belongs to $X(T')$ for each $T'>0$.

In fact, let $u_1,u_2 \in X(T')$ be two solutions of integral equation (15) and let $u=u_1-u_2$, then

$$\begin{aligned}u(x,t) = &-\int_0^t \sin(t-\tau)[u_\tau(x,\tau)+f(u_1(x,\tau))-f(u_2(x,\tau))]\mathrm{d}\tau\\ &+\int_0^t \sin(t-\tau)G*[u(x,\tau)+u_\tau(x,\tau)+f(u_1(x,\tau))-f(u_2(x,\tau))]\mathrm{d}\tau.\end{aligned} \tag{33}$$

$$\begin{aligned}u_t(x,t) = &-\int_0^t \cos(t-\tau)[u_\tau(x,\tau)+f(u_1(x,\tau))-f(u_2(x,\tau))]\mathrm{d}\tau\\ &+\int_0^t \cos(t-\tau)G*[u(x,\tau)+u_\tau(x,\tau)+f(u_1(x,\tau))-f(u_2(x,\tau))]\mathrm{d}\tau.\end{aligned} \tag{34}$$

From the definition of the space $X(T')$, we have $\|u_i(t)\|_\infty \le C_1(T')$ for $i=1,2$ and $0\le t\le T' < T$, where $C_1(T')$ is a constant dependent on T'. Thus, from (33), (34), and Lemmas 2 – 4, we obtain

$$\|u\|_{2,p} \le \int_0^t \|u\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|f(u_1)-f(u_2)\|_{2,p}\mathrm{d}\tau \le C_2(T')\int_0^t \|u\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau, \tag{35}$$

$$\|u_t\|_{2,p} \le \int_0^t \|u\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|f(u_1)-f(u_2)\|_{2,p}\mathrm{d}\tau \le C_2(T')\int_0^t \|u\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau, \tag{36}$$

where $C_2(T')$ is a constant dependent on $C_1(T')$. Combining (35) with (36) yields

$$\|u\|_{2,p}+\|u_t\|_{2,p}\le C_3(T')\int_0^t[\|u\|_{2,p}+\|u_\tau\|_{2,p}]\mathrm{d}\tau, \quad (37)$$

where $C_3(T')$ is a constant dependent on $C_2(T')$. By Gronwall's inequality, we get from (37) that $\|u\|_{2,p}+\|u_t\|_{2,p}=0$ for $0\le t\le T'$. Hence $u=0$ for $0\le t\le T'$, i. e., (15) has at most one solution which belongs to $X(T')$.

Now, let $[0,T_0)$ be the maximal time interval of existence for $u\in X(T_0)$. We want to show that, if (32) is satisfied, then $T_0=\infty$.

Suppose that (32) holds and $T_0=\infty$. For each $T'\in[0,T_0)$, we consider the integral equation

$$\begin{aligned} v(x,t)&=u(x,T')\cos t+u_1(x,T')\sin t\\ &-\int_0^t\sin(t-\tau)\left[v_\tau(x,\tau)+f(v(x,\tau))\right]\mathrm{d}\tau\\ &+\int_0^t\sin(t-\tau)G*\left[v(x,\tau)+v_\tau(x,\tau)+f(v(x,\tau))\right]\mathrm{d}\tau. \end{aligned} \quad (38)$$

By (32),

$$\|u(\cdot,t)\|_{2,p}+\|u_t(\cdot,t)\|_{2,p}+\|u(\cdot,t)\|_\infty+\|u_t(\cdot,t)\|_\infty\le K,$$

where K is a positive constant independent of $T'\in[0,T_0)$. From Lemma 5 and the contraction mapping principle we see that there exists a constant $T_1\in(0,T_0)$, such that, for each $T'\in[0,T_0)$, the integral equation (38) has a unique solution $v(x,t)\in X(T_1)$. In particular, (23) and (30) reveal that T_1 can be selected independently of $T'\in[0,T_0)$. Take $T'=T_0-T_1/2$ and define

$$\tilde{u}(x,t)=\begin{cases}u(x,t), & t\in[0,T'],\\ v(x,t-T'), & t\in[T',T_0+T_1/2],\end{cases}$$

then $\tilde{u}(x,t)$ is a solution of (15) on interval $[0,T_0+T_1/2]$ and, by the uniqueness, $\tilde{u}$ extends u, which violates the maximality of $[0,T_0)$. Therefore, if (32) holds, then $T_0=\infty$. Theorem 1 is proved.

Remark 1. If $u(x,t)\in \mathrm{C}^1([0,T_0);W^{2,p}\cap L^\infty)$ is the solution of (15), from Lemma 2, we know that $u(x,t)\in \mathrm{C}^2([0,T_0);W^{2,p}\cap L^\infty)$ and (14) holds.

3. Existence and Uniqueness of the Global Solution

In this section, we prove the existence and the uniqueness of the global solutions for problem (1) and (2). For this purpose we are going to make a priori estimates of the local solutions for problem (1) and (2).

Lemma 6. Suppose that $f(u)\in C(\mathrm{R})$, $F(u)=\int_0^u f(s)\mathrm{d}s$, $(-\Delta)^{-1/2}u_1\in L^2$, $u_0,u_1\in L^2$, and $F(u_0)\in L^1$, then for the solution $u(x,t)$ of problem (1) and (2), we have the energy identity

$$\begin{aligned} E(t)&=\|(-\Delta)^{-1/2}u_t\|_2^2+\|u_t\|_2^2+\|u\|_2^2\\ &+2\int_0^t\|u_\tau\|_2^2\mathrm{d}\tau+2\int_{\mathrm{R}^n}F(u)\mathrm{d}x=E(0). \end{aligned} \quad (39)$$

Here and in the sequel $(-\Delta)^{-\alpha}u(x)=\mathcal{F}^{-1}[|x|^{-2\alpha}\mathcal{F}u(x)]$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, Fourier transformation and inverse Fourier transformation in R^n (see [20]).

Proof. Multiplying (1) by $(-\Delta)^{-1}u_t$ and integrating the product with respect to x, we obtain

$$(u_{tt}-\Delta u-\Delta u_{tt}-\Delta u_t-\Delta f(u),(-\Delta)^{-1}u_t)=0,$$

$$\Big((-\Delta)^{-1}u_{tt}+u+u_{tt}+u_t+f(u),u_t\Big)=0,$$

$$\Big((-\Delta)^{-1/2}u_{tt},(-\Delta)^{-1/2}u_t\Big)+(u,u_t)+(u_{tt},u_t)+(u_t,u_t)+\big(f(u),u_t\big)=0.$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|(-\Delta)^{-1/2}u_t\|_2^2+\|u_t\|_2^2+\|u\|_2^2+2\int_{\mathrm{R}^n}F(u)\mathrm{d}x\right]+\|u_t\|_2^2=0,$$

where $(\cdot,\cdot)$ denotes the inner product of L^2 space. Integrating the above equality with respect to t over $[0,t]$, we get (39). The lemma is proved.

Lemma 7. Assume that $f(u)\in C(\mathrm{R})$, $F(u)=\int_0^u f(s)\mathrm{d}s$, $(-\Delta)^{-1/2}u_1\in L^2$, $F(u)\ge 0$, $u_0,u_1\in L^2\cap L^\infty$, and $F(u_0)\in L^1$. If there exists ρ that satisfies

$$\begin{aligned} &1\le\rho\le\infty, \text{ if } n=1,\\ &1<\rho\le\infty, \text{ if } n=2,\\ &\frac{n}{2}<\rho\le\infty, \text{ if } n\ge 3, \end{aligned} \quad (40)$$

such that

$$|f(u)|\le A(F(u))^{1/\rho}|u|+B, \quad (41)$$

where A and B are positive constants, then the solution $u(x,t)$ of problem (1) and (2) has the estimation

$$\|u_t\|_\infty^2+\|u\|_\infty^2\le C_1(T),\quad 0\le t\le T. \quad (42)$$

Here and in the sequel $C_i(T)$ $(i=1,2,\ldots)$ are constants dependent on T.

Example 1. $f(u) = u^{2k+1}$ and $\rho = 1 + \frac{1}{k}$ satisfy the hypotheses if $0 \le k < \infty$ for $n = 1,2$ or if $0 \le k < 2$ for $n = 3$. Obviously, when $k = 1$, $\rho = 2$, and $1 \le n \le 3$ the nonlinear term u^3 of (5) satisfies inequality (41).

Proof. Multiplying both sides of (14) by u_t yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[u_t^2 + u^2 + 2F(u)] + 2u_t^2 \\ &= 2(G*u)u_t + 2(G*u_t)u_t + 2(G*f(u))u_t.\end{aligned} \tag{43}$$

We make use of inequality (41), (12) and Young's inequality to get

$$\begin{aligned}|G*f(u)| &\le A[G*(F(u))^{1/\rho}|u|] + |G*B| \\ &\le A\|G\|_q\|(F(u))^{1/\rho}u\|_\rho + B \\ &\le A\|G\|_q\|u\|_\infty\|(F(u))\|_1^{1/\rho} + B,\end{aligned} \tag{44}$$

where $\frac{1}{\rho} + \frac{1}{q} = 1$. By using (40), (12), Lemma 4, and Lemma 6, we have from (44)

$$\begin{aligned}&|G*f(u)| \le C_2(T)\|u\|_\infty + B, \\ &|G*u| \le \|u\|_\infty, \\ &|G*u_t| \le \|u_t\|_\infty.\end{aligned}$$

Substituting the above inequalities into (43) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[u_t^2 + u^2 + 2F(u)] + 2u_t^2 \\ &\le C_3(T)\|u\|_\infty\|u_t\|_\infty + 2B\|u_t\|_\infty + 2\|u_t\|_\infty^2,\end{aligned} \tag{45}$$

integrating (45) with respect to t and using the Cauchy inequality, we get

$$\begin{aligned}&\|u_t\|_\infty^2 + \|u\|_\infty^2 + 2\|F(u)\|_\infty \\ &\le \|u_1\|_\infty^2 + \|u_0\|_\infty^2 + 2\|F(u_0)\|_\infty + BT \\ &\quad + C_4(T)\int_0^t (\|u_\tau(\tau)\|_\infty^2 + \|u(\tau)\|_\infty^2)\mathrm{d}\tau, \\ &0 \le t \le T.\end{aligned} \tag{46}$$

Since $F(u) \ge 0$, it follows from Gronwall's inequality and inequality (46) that inequality (42) can be obtained. The lemma is proved.

Lemma 8. Assume that $f(u) \in C^3(\mathrm{R})$, $F(u) = \int_0^u f(s)\mathrm{d}s$, $(-\Delta)^{-1/2}u_1 \in L^2$, $F(u) \ge 0$, $u_0, u_1 \in W^{2,p} \cap L^2 \cap L^\infty$, and $F(u_0) \in L^1$, then the solution $(u(x,t)$ of problem (1) and (2) has the estimation

$$\|u\|_{2,p} + \|u_t\|_{2,p} \le C_5(T), \quad 0 \le t \le T. \tag{47}$$

Proof. Using Young's inequality, we get from (15), Lemma 2, Lemma 3 and Lemma 4

$$\begin{aligned}&\|u\|_{2,p} \le \|u_0\|_{2,p} + \|u_1\|_{2,p} + \int_0^t \|u_\tau + f(u)\|_{2,p}\mathrm{d}\tau \\ &+ \int_0^t \|G*[u + u_\tau + f(u)]\|_{2,p}\mathrm{d}\tau \le \|u_0\|_{2,p} + \|u_1\|_{2,p} \\ &+ \int_0^t \|u\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|f(u)\|_{2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau \\ &\le \|u_0\|_{2,p} + \|u_1\|_{2,p} + C_6(T)\int_0^t \|u\|_{2,p}\mathrm{d}\tau \\ &\quad + 2\int_0^t \|u_\tau]\|_{2,p}\mathrm{d}\tau.\end{aligned} \tag{48}$$

Integrating (14) with respect to t, we have

$$\begin{aligned}u_t = u_1(x) &- \int_0^t [u + u_\tau + f(u)]\mathrm{d}\tau \\ &+ \int_0^t G*[u + u_\tau + f(u)]\mathrm{d}\tau.\end{aligned} \tag{49}$$

By use of Young's inequality, we get from (49) and Lemmas 2 – 4

$$\begin{aligned}\|u_t\|_{2,p} \le \|u_1\|_{2,p} &+ C_7(T)\int_0^t \|u\|_{2,p}\mathrm{d}\tau \\ &+ 2\int_0^t \|u_\tau\|_{2,p}\mathrm{d}\tau.\end{aligned} \tag{50}$$

Combining (48) with (50) yields

$$\begin{aligned}&\|u\|_{2,p} + \|u_t\|_{2,p} \le \|u_0\|_{2,p} + 2\|u_1\|_{2,p} \\ &+ C_8(T)\int_0^t (\|u\|_{2,p} + \|u_\tau\|_{2,p})\mathrm{d}\tau.\end{aligned} \tag{51}$$

By use of Gronwall's inequality, we get from (51) $\|u\|_{2,p} + \|u_t\|_{2,p} \le C_5(T)$, i. e. inequality (47) holds. The lemma is proved.

Theorem 2. Assume that $f(u) \in C^3(\mathrm{R})$ satisfies conditions (40), (41), $F(u) = \int_0^u f(s)\mathrm{d}s$, $(-\Delta)^{-1/2}u_1 \in L^2$, $F(u) \ge 0$, $u_0, u_1 \in W^{2,p} \cap L^2 \cap L^\infty$, and $F(u_0) \in L^1$, then problem (1) and (2) admits a unique global solution $u(x,t) \in \mathrm{C}^2([0,\infty); W^{2,p} \cap L^2 \cap L^\infty)$ and $(-\Delta)^{-1/2})u_t \in L^2$.

Proof. From Theorem 1 and Remark 1 it follows that problem (1) and (2) has a unique local solution $u(x,t) \in \mathrm{C}^2([0,T_0); W^{2,p} \cap L^\infty)$. And from Lemmas 6 – 8, we know that problem (1) and (2) has a unique global solution $u(x,t) \in \mathrm{C}^2([0,\infty); W^{2,p} \cap L^\infty)$ and $(-\Delta)^{-1/2})u_t \in L^2$. The theorem is proved.

Theorem 3. Assume that $f(u) \in C^{k+3}(\mathrm{R})$, where $k \geq 0$ is an arbitrary integer, satisfies conditions (40), (41), $F(u) = \int_0^u f(s)\mathrm{d}s$, $(-\Delta)^{-1/2}u_1 \in L^2$, $F(u) \geq 0$, $u_0, u_1 \in W^{k+2,p} \cap L^2 \cap L^\infty$, and $F(u_0) \in L^1$, then for any $T > 0$, problem (1) and (2) admits a unique solution $u(x,t) \in \mathrm{C}^2([0,T]; W^{k+2,p} \cap L^2 \cap L^\infty)$ and $(-\Delta)^{-1/2})u_t \in L^2$.

Proof. We make use of a similar method as in Theorem 2. In the proof of the local existence we define the function space

$$X_k(T) = \mathrm{C}^1([0,T]; W^{k+2,p} \cap L^\infty$$

furnished with the norm

$$\|u\|_{X_k(T)} = \max_{0\leq t\leq T} \|u\|_{k+2,p} + \max_{0\leq t\leq T} \|u_t\|_{k+2,p} + \max_{0\leq t\leq T} \|u\|_\infty + \max_{0\leq t\leq T} \|u_t\|_\infty, \quad \forall u \in X_k(T)$$

instead of $X(T)$, and we need to consider a metric space $Y_k(M,T)$ instead of $Y(M,T)$, where

$$Y_k(M,T) = \{u | u \in X_k(T), \|u\|_{X_k(T)} \leq 2M+1\}.$$

By a similar way used to prove Theorem 1 and from Remark 1, one can easily see that problem (1) and (2) has a unique solution $u(x,t) \in \mathrm{C}^2([0,T_0]; W^{k+2,p} \cap L^\infty)$. By the similar way to get inequality (47), we obtain

$$\|u\|_{k+2,p} \leq \|u_0\|_{k+2,p} + \|u_1\|_{k+2,p} + C_9(T)\int_0^t \|u\|_{k+2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{k+2,p}\mathrm{d}\tau, \tag{52}$$

$$\|u_t\|_{k+2,p} \leq \|u_1\|_{k+2,p} + C_{10}(T)\int_0^t \|u\|_{k+2,p}\mathrm{d}\tau + 2\int_0^t \|u_\tau\|_{k+2,p}\mathrm{d}\tau. \tag{53}$$

Combining (52) with (53) yields

$$\|u\|_{k+2,p} + \|u_t\|_{k+2,p} \leq \|u_0\|_{k+2,p} + 2\|u_1\|_{k+2,p} + C_{11}(T)\int_0^t (\|u\|_{k+2,p} + \|u_\tau\|_{k+2,p})\mathrm{d}\tau. \tag{54}$$

By use of Gronwall's inequality, we get from (54)

$$\|u\|_{k+2,p} + \|u_t\|_{k+2,p} \leq C_{12}(T). \tag{55}$$

Thus, from Lemmas 6–8 and (55), it follows that problem (1) and (2) has a unique solution $u(x,t) \in \mathrm{C}^2([0,T]; W^{k+2,p} \cap L^2 \cap L^\infty)$ $(\forall T > 0)$ and $(-\Delta)^{-1/2}u_t \in L^2$. The theorem is proved.

4. Blow up of the Solution

In this section, we are going to consider the blow up of the solution of problem (1) and (2) by the concavity method. For this purpose, we give the following lemma [11] which is a generalization of Levine's result [12].

Lemma 9. Suppose that a positive, twice differentiable function $\psi(t)$ satisfies on $t \geq 0$ the inequality

$$\psi''(t)\psi(t) - (1+\upsilon)(\psi'(t))^2 \geq -2M_1\psi(t)\psi'(t) - M_2(\psi(t))^2,$$

where $\upsilon > 0$ and $M_1, M_2 \geq 0$ are constants. If $\psi(0) > 0$, $\psi'(0) > -\gamma_2\upsilon^{-1}\psi(0)$, and $M_1 + M_2 > 0$, then $\psi(t)$ tends to infinity as

$$t \to t_1 \leq t_2 = \frac{1}{2\sqrt{M_1^2 + \upsilon M_2}} \ln \frac{\gamma_1\psi(0) + \upsilon\psi'(0)}{\gamma_2\psi(0) + \upsilon\psi'(0)},$$

where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \upsilon M_2}$. If $\psi(0) > 0$, $\psi'(0) > 0$ and $M_1 = M_2 = 0$, then $\psi(t) \to \infty$ as $t \to t_1 \leq t_2 = \psi(0)/\upsilon\psi'(0)$.

Theorem 4. Assume that $f(u) \in C(\mathrm{R})$, $u_0, u_1 \in L^2$, $(-\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1 \in L^2$, $F(u) = \int_0^u f(s)\mathrm{d}s$, $F(u_0) \in L^1$, and there exists a constant $\alpha > 0$ such that

$$2f(u)u \leq (6+2\alpha)F(u) + \alpha u^2, \quad \forall u \in \mathrm{R}. \tag{56}$$

Then the solution $u(x,t)$ of problem (1) and (2) blows up in finite time if one of the following conditions is valid:

(i) $E(0) = \|(-\Delta)^{-1/2}u_1\|_2^2 + \|u_1\|_2^2 + \|u_0\|_2^2 + 2\int_{\mathrm{R}^n} F(u_0)\mathrm{d}x < 0$,

(ii) $E(0) = 0$ and $((\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1) + (u_0, u_1) > 0$,

(iii) $E(0) > 0$ and $\left((\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1\right) + (u_0, u_1) > \sqrt{\frac{\alpha+3}{\alpha+2}E(0)(\|(-\Delta)^{-1/2}u_0\|_2^2 + \|u_0\|_2^2)}$.

Proof. Suppose that the maximal time of existence of the solution of problem (1) and (2) is infinite. A contradiction will be obtained by Lemma 9. Let

$$\psi(t) = \|(-\Delta)^{-1/2}u\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2, \tag{57}$$

where β and τ are nonnegative constants to be specified later. Obviously we have

$$\psi'(t) = 2\big[\big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_t\big) + (u,u_t) + \beta(t+\tau)\big]. \quad (58)$$

Using the Schwarz inequality and the inequality,

$$(a_1b_1 + \ldots + a_nb_n)^2 \le (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2),$$

where $a_i, b_i \ge 0$, $i = 1, \ldots, n$, we have

$$\begin{aligned}(\psi'(t))^2 &\le 4\big[\|(-\Delta)^{-1/2}u\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2\big] \\ &\quad \cdot \big[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta\big] \\ &= 4\psi(t)\big[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta\big].\end{aligned} \quad (59)$$

We get from (1)

$$\begin{aligned}\psi''(t) &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 \\ &\quad + 2((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_{tt}) + 2(u,u_{tt}) + 2\beta \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta_0 \\ &\quad + 2(u, (-\Delta)^{-1}u_{tt} + u_{tt}) \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta \\ &\quad - 2(u, u + u_t + f(u)) \\ &= 2\|(-\Delta)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta \\ &\quad - 2\|u\|_2^2 - 2(u,u_t) - 2\int_{\mathrm{R}^n} uf(u)\mathrm{d}x.\end{aligned} \quad (60)$$

By the aid of the Cauchy inequality and equality (39) we have

$$\begin{aligned}2(u,u_t) &\le \|u\|_2^2 + \|u_t\|_2^2 = E(0) - \|(-\Delta)^{-1/2}u_t\|_2^2 \\ &\quad -2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau - 2\int_{\mathrm{R}^n} F(u)\mathrm{d}x.\end{aligned} \quad (61)$$

From (57) – (61) we obtain

$$\begin{aligned}&\psi(t)\psi''(t) - \Big(1+\frac{\alpha}{4}\Big)(\psi'(t))^2 \ge \psi(t)\psi''(t) \\ &-(4+\alpha)\psi(t)\big[\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta\big] \\ &\ge \psi(t)\Big\{(-1-\alpha)\|(-\Delta)^{-1/2}u_t\|_2^2 + (-2-\alpha)\|u_t\|_2^2 \\ &+(-2-\alpha)\beta + \int_{\mathrm{R}^n}[F(u) - 2uf(u) - 2u^2]\mathrm{d}x \\ &+2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau - E(0)\Big\}.\end{aligned} \quad (62)$$

From equality (39) we have

$$\begin{aligned}&(-1-\alpha)\|(-\Delta)^{-1/2}u_t\|_2^2 + (-2-\alpha)\|u_t\|_2^2 \\ &\ge (-2-\alpha)(\|(-\Delta)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2) \\ &= (2+\alpha)(\|u\|_2^2 + 2\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau + 2\int_{\mathrm{R}^n} F(u)\mathrm{d}x - E(0)).\end{aligned}$$

Thus, from the above inequality, inequalities (56) and (62), we get

$$\begin{aligned}&\psi(t)\psi''(t) - \Big(1+\frac{\alpha}{4}\Big)(\psi'(t))^2 \ge \psi(t)\Big\{-(2+\alpha)\beta \\ &-(3+\alpha)E(0) + \int_{\mathrm{R}^n}[(6+2\alpha)F(u) + \alpha u^2 - 2uf(u)]\mathrm{d}x \\ &+(6+2\alpha)\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau\Big\} \\ &\ge -[(2+\alpha)\beta + (3+\alpha)E(0)]\psi(t).\end{aligned} \quad (63)$$

If $E(t) < 0$, taking $\beta = -\frac{3+\alpha}{2+\alpha}E(0) > 0$, then

$$\psi(t)\psi''(t) - \Big(1+\frac{\alpha}{4}\Big)(\psi'(t))^2 \ge 0.$$

We may now choose τ as large that $\psi'(0) > 0$. From Lemma 9 we know that $\psi(t)$ becomes infinite at a time T_1 at most equal to

$$T_2 = \frac{4\psi(0)}{\alpha\psi'(0)} < \infty.$$

If $E(0) = 0$, taking $\beta = 0$, then we get from (63)

$$\psi(t)\psi''(t) - \Big(1+\frac{\alpha}{4}\Big)(\psi'(t))^2 \ge 0.$$

Also $\psi'(0) > 0$ by assumption (ii). Thus, we obtain from Lemma 9 that $\psi(t)$ becomes infinite at a time T_1 at most equal to

$$T_2 = \frac{4\psi(0)}{\alpha\psi'(0)} < \infty.$$

If $E(0) > 0$ and taking $\beta = 0$, inequality (63) becomes

$$\begin{aligned}&\psi(t)\psi''(t) - \Big(1+\frac{\alpha}{4}\Big)(\psi'(t))^2 \\ &\ge -(3+\alpha)E(0)\psi(t).\end{aligned} \quad (64)$$

Define $J(t) = (\psi(t))^{-\upsilon}$, where $\upsilon = \alpha/4$. Then

$$\begin{aligned}J'(t) &= -\upsilon(\psi(t))^{-\upsilon-1}\psi'(t), \\ J''(t) &= -\upsilon(\psi(t))^{-\upsilon-2}[\psi(t)\psi''(t) - (1+\upsilon)(\psi'(t))^2] \\ &\le \upsilon(3+4\upsilon)E(0)(\psi(t))^{-\upsilon-1},\end{aligned} \quad (65)$$

where inequality (64) is used. Assumption (iii) implies that $J'(0) < 0$. Let

$$t^* = \sup\{t \mid J'(\tau) < 0,\ \tau \in (0,t)\}. \quad (66)$$

By the continuity of $J'(t)$, t^* is positive. Multiplying (65) by $2J'(t)$ yields

$$[(J'(t))^2]' \geq -2\upsilon^2(3+4\upsilon)E(0)(\psi(t))^{-2\upsilon-2}\psi'(t) = 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)[(\psi(t))^{-2\upsilon-1}]', \quad (67)$$

$\forall t \in [0,t^*)$.

Integrate (67) with respect to t over $[0,t)$ to get

$$(J'(t))^2 \geq (J'(t))^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)(\psi(t))^{-2\upsilon-1} \geq (J'(0))^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)(\psi(0))^{-2\upsilon-1}.$$

By assumption (iii)

$$(J'(0))^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)(\psi(0))^{-2\upsilon-1} > 0.$$

Hence by continuity of $J'(t)$, we obtain

$$J'(t) \leq -\Big[(J'(0))^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)(\psi(0))^{-2\upsilon-1}\Big]^{1/2} \quad (68)$$

for $0 \leq t < t^*$. By the continuity of t^*, it follows that inequality (68) holds for all $t \geq 0$. Therefore

$$J(t) \leq J(0) - \Big[(J'(0))^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0) \cdot (\psi(0))^{-2\upsilon-1}\Big]^{1/2} t,\ \forall t > 0.$$

So $J(T_1) = 0$ for some T_1 and

$$0 < T_1 \leq T_2 = J(0)/\big[(J'(0))^2 - [\alpha^2(3+\alpha)/(4\alpha+8)] \cdot E(0)(\psi(0))^{-(\alpha+2)/2}\big]^{1/2}.$$

Thus $\psi(t)$ becomes infinite at time T_1.

Therefore $\psi(t)$ becomes infinite at time T_1 under either assumptions (i), (ii) or (iii). We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof.

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